

“On the Impossibility of a Poincaré–Invariant Vacuum State with Unit Norm” Refuted

Stephen J. Summers

Department of Mathematics
University of Florida
Gainesville FL 32611, USA

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Abstract

Assertions made in a document recently deposited in the arXiv are refuted.

Recently, a literally incredible document was deposited in the arXiv [3]. I shall restrict my comments to a direct rebuttal of the author’s central point, which is indicated in the title of the document. The author claims “The axioms of QFT cannot be made internally consistent.” This conclusion is arrived at on the grounds that “It is impossible for any state to be both Poincaré–invariant and also have unit norm.”

For most readers, it will suffice to be reminded that models satisfying “the axioms of QFT”¹ have been constructed with full mathematical rigor by a number of techniques and approaches [1, 4, 5, 7, 8, 10, 13].² In these models the state which the author claims cannot exist *does*, in fact, exist. However, if there is anyone remaining who doubts that the author’s reasoning must be faulty, I shall point out some of the errors in his “proof” that “It is impossible for any state to be both Poincaré–invariant and also have unit norm.” These errors obviate his conclusion.

For the reader’s convenience, I shall review the author’s argument. However, I shall use more standard and consistent notation where necessary. He begins with a separable and infinite dimensional Hilbert space \mathcal{H} and an unspecified orthonormal basis $\{\Psi_n\}_{n \in \mathbb{N}}$ for \mathcal{H} . He takes a vector $\Psi \in \mathcal{H}$ and expands it with respect to the given basis:

$$\Psi = \sum_{n=1}^{\infty} c_n \Psi_n. \quad (0.1)$$

¹From the context, the author is referring to the Wightman axioms [11].

²This list of references is far from exhaustive.

Then the problems begin.

To proceed, a few noncontroversial clarifying remarks will be useful. The translation subgroup of the isometry group of four dimensional Minkowski space is usually realized as the additive group of four dimensional real vectors, denoted here by \mathbb{R}^4 . For well known, physically motivated reasons, this symmetry group acts upon the Hilbert space of states solely through the intermediary of a unitary representation $U(\mathbb{R}^4)$ [2, 4, 11, 12]. Hence, the phrase “a vector $\Psi \in \mathcal{H}$ is translation-invariant” has the following mathematical meaning:

$$U(a)\Psi = \Psi, \quad (0.2)$$

for all $a \in \mathbb{R}^4$.

Here comes the rub: the author, without a word of justification, claims the following is true:

$$U(a) \sum_{n=1}^{\infty} c_n \Psi_n = \sum_{n=1}^{\infty} c_{n+a} \Psi_n, \quad (0.3)$$

for all $a \in \mathbb{R}^4$. Then, for a translation-invariant vector it follows from equations (0.1), (0.2) and (0.3) that

$$\sum_{n=1}^{\infty} c_n \Psi_n = \sum_{n=1}^{\infty} c_{n+a} \Psi_n,$$

and the orthonormality of the basis then yields

$$c_n = c_{n+a}, \quad (0.4)$$

for all $n \in \mathbb{N}$ (in particular, for $n = 1$) and $a \in \mathbb{R}^4$. He then (tacitly) lets a run through the natural numbers (!!!) to conclude $c_1 = c_n$ for all $n \in \mathbb{N}$. Of course, since for any vector $\Psi \in \mathcal{H}$ one must have $\lim_{n \rightarrow \infty} c_n = 0$, this yields $\Psi = 0$.³ The author therefore “proves” that the only translation-invariant vector in \mathcal{H} is the zero vector.

Doubtless, most readers have recognized equation (0.3) and the argument after (0.4) to be nonsensical, but just in case someone misses the point, I shall explain.

(1) Even if the equation made sense, (0.3) does not represent the action of the translation group in any known quantum field model — cf. any of the examples cited above — and therefore it is certainly not entailed by the Wightman axioms. To emphasize the point: equation (0.3) is purely the author’s fantasy, which he does not even try to support.

(2) However, equation (0.3) is not even mathematically meaningful. Since $n \in \mathbb{N}$ and $a \in \mathbb{R}^4$, what does $n + a$ mean? What does c_{n+a} mean if $n + a$ is not a natural number? Has it become necessary to point out the fact that if one allows meaningless quantities into an argument, then *anything* can be “proven”, including $1 = 2$?

³Note that $\Psi \in \mathcal{H}$ entails $\|\Psi\|$ must be finite.

(3) In addition, I point out the following facts: The mapping

$$V \sum_{n=1}^{\infty} c_n \Psi_n = \sum_{n=1}^{\infty} c_{n+k} \Psi_n$$

(which *does* make sense when $k \in \mathbb{N}$ and to which⁴ the author is tacitly appealing when he employs the argument after (0.4)) depends upon the choice of the basis $\{\Psi_n\}_{n \in \mathbb{N}}$ (of which there are uncountably infinitely many in a separable Hilbert space) and is not unitary, for any choice of $k \in \mathbb{N}$. For these and many other reasons which need not be detailed here, such operators have no physical interpretation in quantum field theory. And, what is more to the point, they have nothing to do with the spacetime symmetry translations the Wightman axioms actually refer to.

References

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⁴This operator is the k th power of the adjoint of the unilateral shift operator well known to functional analysts, and it has long been known that the powers of this adjoint tend to the zero operator in the strong operator topology, cf. [6, 9].